Problem Set 1 – Relativity (G0I36A)

22/09/2020

1 Basics of Index Notation

In this problem, we'll get a feel for the basic rules of index notation. Remember to always keep in mind what the free indices and what the dummy indices are in all expressions.

1.) Which of the following expressions are allowed? Which are not? For the ones that are allowed, how many components does the object have?

   a.) $T_{\nu}{}^{\mu}$
   b.) $T^{\mu\mu}$
   c.) $T^{\mu}{}_{\nu}T_{\nu\mu}$
   d.) $T^{\mu\nu}T_{\nu}{}^{\rho}$

2.) Which of the following equations are allowed? Which are not? For the ones that are allowed, how many components does the equation have?

   a.) $T^{\mu}{}_{\nu}A^{\nu} = A^{\mu}$
   b.) $T_{\mu\nu}A^{\nu} = A^{\mu}$
   c.) $T^{\mu}{}_{\mu}A^{\nu} = A^{\nu}$
   d.) $T^{\mu}{}_{\nu}A^{\nu} = T^{\rho}{}_{\nu}A^{\nu}$
   e.) $T^{\mu}{}_{\nu}A^{\nu}B_{\nu} = C^{\mu}$

3.) Consider the equation $T_{\mu\nu}A^{\mu}B^{\nu} = T_{\rho\sigma}A^{\rho}B^{\sigma}$. Is this a valid equation? If it is not allowed, why not? If it is allowed, does this equation tell us anything useful?

2 Index Gymnastics

Ok, the first problem was just a warmup. Let's move on to some harder problems as well as a few proofs and useful results. You may find the identities we prove in this section useful throughout the rest of the class, so don't skip them!

4.) Let $T^{\mu\nu}$ be a $(2,0)$ tensor. Write down what $T_{\mu}{}^{\nu}$, $T^{\mu}{}_{\nu}$, and $T_{\mu\nu}$ are using the metric tensor. Then, give explicit formulas for T_{00}, T_{10}, $T_{0}{}^{1}$, $T_{1}{}^{1}$, and $T^{1}{}_{1}$.

5.) Show that $A_{\mu}B^{\mu} = A^{\mu}B_{\mu}$ for any vectors A^{μ} and B^{μ}.

6.) **Bonus:** Prove that for two $(n,0)$ tensors $T^{\mu_1\cdots\mu_n}$ and $S^{\mu_1\cdots\mu_n}$, any pair of contracted indices between them can be raised and lowered freely:

$$T^{\mu_1\cdots\mu_{i-1}\rho\,\mu_{i+1}\cdots\mu_n}S^{\nu_1\cdots\nu_{i-1}}{}_{\rho}{}^{\nu_{i+1}\cdots\nu_n} = T^{\mu_1\cdots\mu_{i-1}}{}_{\rho}{}^{\mu_{i+1}\cdots\mu_n}S^{\nu_1\cdots\nu_{i-1}\rho\,\nu_{i+1}\cdots\nu_n}\ . \tag{2.1}$$

7.) For any tensor $T^{\mu\nu}$, we can define its symmetric piece $T^{(\mu\nu)}$ and its antisymmetric piece $T^{[\mu\nu]}$ by the following:

$$\begin{aligned} T^{(\mu\nu)} &\equiv \frac{1}{2}\left(T^{\mu\nu} + T^{\nu\mu}\right)\ , \\ T^{[\mu\nu]} &\equiv \frac{1}{2}\left(T^{\mu\nu} - T^{\nu\mu}\right)\ . \end{aligned} \tag{2.2}$$

Show that $T^{\mu\nu} = T^{(\mu\nu)} + T^{[\mu\nu]}$.

8.) Prove that $T^{[\mu\nu]}S_{(\mu\nu)} = 0$ for any tensors $T^{\mu\nu}$ and $S^{\mu\nu}$.

9.) Prove that $T_{\mu\nu}A^\mu A^\nu = T_{(\mu\nu)}A^\mu A^\nu$ for any tensor $T^{\mu\nu}$ and vector A^μ.

10.) We can define the trace T of $T^{\mu\nu}$ by

$$T \equiv \eta_{\mu\nu}T^{\mu\nu} \ . \tag{2.3}$$

In a four-dimensional spacetime, the trace-free version of $T^{\mu\nu}$ is then defined by

$$\hat{T}^{\mu\nu} \equiv T^{\mu\nu} - \frac{1}{4}\eta^{\mu\nu}T \ . \tag{2.4}$$

Prove that the trace of $\hat{T}^{\mu\nu}$ is in fact zero.

11.) **Bonus:** If we are now in a d-dimensional spacetime (where d is not necessarily four), how do we generalize the trace-free $\hat{T}^{\mu\nu}$? That is, what is the generalization of equation (2.4) such that $\hat{T}^{\mu\nu}$ is trace-free?

3 Levi-Civita Symbol

Another important tensor that will play a role in this class is the Levi-Civita tensor $\varepsilon_{\mu\nu\rho\sigma}$. It is a $(0, 4)$ tensor that is defined as follows:

$$\varepsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \{\mu,\nu,\rho,\sigma\} \text{ is an even permutation of } \{0,1,2,3\} \\ -1 & \text{if } \{\mu,\nu,\rho,\sigma\} \text{ is an odd permutation of } \{0,1,2,3\} \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

Notice that this definition implies that $\varepsilon_{\mu\nu\rho\sigma} = 0$ if any of its indices take the same value.

12.) Prove that for any tensor $T^{\mu\nu}$, the following identity is true:

$$\varepsilon_{\mu\nu\rho\sigma}T^{\rho\sigma} = \varepsilon_{\mu\nu\rho\sigma}T^{[\rho\sigma]} \ . \tag{3.2}$$

As a corollary of this result, show that

$$\varepsilon_{\mu\nu\rho\sigma}\eta^{\rho\sigma} = 0 \ . \tag{3.3}$$

What if we were to change which two indices are contracted? That is, what can you say about $\varepsilon_{\mu\nu\rho\sigma}T^{\nu\sigma}$, $\varepsilon_{\mu\nu\rho\sigma}T^{\mu\nu}$, etc.?